Concepts of Special Relativity

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Preamble

This text presents the main basic kinematics concepts of Special Relativity, namely, time dilation, length contraction and the combination law for collinear velocities. The popular "twin paradox" is also discussed.

The approach employed was developed by the British cosmologist Hermann Bondi and is known in the literature as "k calculus", for it makes use, as basic quantity to specify the relationship between two observers in relative motion, of an adimensional factor for which Bondi adopted the notation k. This factor is nothing more than the factor characteristic of the Doppler effect for light, that is, the ratio between the period of emission of light signals, by one of the observers, and the corresponding period of reception by the other. Bondi showed that the use of this factor in the analysis of specific situations involving the exchange of light pulses between two, or more, observers affords a particularly simple deduction of the, a priori counterintuitive, effects implied by the postulates of Special Relativity.

In the present work, each relevant situation is presented and analysed successively in a generic (algebraic) mathematical notation. In an appendix, concrete cases are considered for which numerical results are obtained. Diagrams for the visualization of the involved motions in space-time, known as Minkowski diagrams, are used to illustrate the generic situations and the specific examples as well.

This text accompanies a set of computerized animations which allow, for each situation, the visualization on the screen of the motions of the various observers, light pulses and other bodies, together with the simultaneous dynamical construction of the Minkowski diagram and of a table registering the occurrence times of the relevant events.

Chapter 1

A few basic concepts

1.1 Introduction

This chapter defines a few basic concepts, which are indispensable for the development of relativistic kinematics and will be used repeatedly in all the arguments. It also introduces a graphical representation of the motions, which will facilitate the visualization of the situations considered for establishing the main consequences of the principles of Special Relativity.

1.2 Event

An ***event*** is something that happens in some specific place at some specific time. Your birth, or the emission of a flash by a photographic camera, are examples of events. Evidently, one is dealing, in most cases, with an idealization, valid if the phenomenon in question had a brief duration and happened in a region of small spatial extension.

1.3 Observer

An ***observer*** is somebody who observes and describes events. Distinct observers may describe the same event in different manners. It should be emphasized that one is not referring here to any subjective difference, but rather to differences that can be rationally explained in terms of the states of motion of the observers.

1.4 Referential

In order to describe an event with precision, an observer utilizes a reference system, or ***referential***. A referential consists of a position coordinate system, employed to specify where the event takes place, and of a time scale, used to stipulate the instant at which the event occurs. Positions can be determined with a metric tape (to measure distances) and a theodolite (to measure angles). Times can be measured with a clock. Thus, concretely, one may associate a referential to an observer equipped with a metric tape, a theodolite and a clock. In general, distinct observers will attribute to the same event different positions and different times.

1.5 Space-time

In the study of Relativity, it is frequently convenient to join the (three-dimensional) position space together with the (one-dimensional) time scale to form a single (four-dimensional) space called ***space-time***. To every event, is associated a point in space-time. The history of any localized entity may be considered as a continuous succession of events. Therefore, such history is represented in space-time by a continuous line, known as the entity's ***world line***.

1.6 Minkowski diagram

The space-time of events and the world lines of physical entities may be visualized with the help of a ***Minkowski diagram***. It is not possible to draw in four dimensions and therefore one must "forget" at least one of the position directions in order to draw a Minkowski diagram. Given that a sheet of paper, or the screen of a computer, are two-dimensional surfaces, more clarity in the drawing is achieved if one can "forget" two spatial directions. The condition for this to be possible is that all events considered happen on the ***same straight line*** of position space. This will be the case for all situations considered in this text.

In a Minkowski diagram, an event is represented by a point. Defining in the diagram a system of orthogonal axes, the event's occurrence time is measured along one of the axes and the event's position is measured along the other axis. In this text, we stipulate that the occurrence time is given in abscissa (horizontal axis on the computer screen) and the position is given in ordinate (vertical axis on the screen). We should emphasize that ***this convention is arbitrary and many Relativity textbooks, as well as research articles, make use of the opposite convention***. The convention adopted here offers the advantage of corresponding to the usual graphic representation of a function, in which the values of the variable are indicated in abscissa and the corresponding values of the function in ordinate. In a Minkowski diagram, the world line of an entity shows the variation of the entity's position as a function of time.

As will be seen in the presentation of the postulates of Special Relativity, the velocity of light in vacuum plays an important role in the theory, namely that of a fundamental universal constant. For this reason, it is convenient to use a unit of distance related to the unit of time in such a way that *the velocity of light be equal to one*. One example of such a distance unit, ordinarily used in Astronomy, is the ***light-year***, which is the distance traveled by light in a year. Obviously then, the velocity of light is one light-year per year. With such units, during a unit time interval, a light pulse moves over a unit distance. Consequently, *the world lines of light pulses are straight lines inclined at 45 degrees in a Minkowski diagram*. According to Special Relativity, no signal or body can move with velocity above the speed of light. Therefore, in a Minkowski diagram, no world line can have an inclination superior to 45 degrees with respect to the horizontal axis. As a consequence, the width of such diagrams will usually be greater than their height. One should stress that this characteristic results from the choice of the horizontal direction for the time axis and constitutes an additional motivation for adopting that convention: the present text accompanies an animation program and a computer screen is, at least in its usual orientation, wider than it is high.

Since the space and time coordinates depend on the referential used, one always needs to specify which observer is measuring the values indicated in abscissa and in ordinate in a Minkowski diagram. .

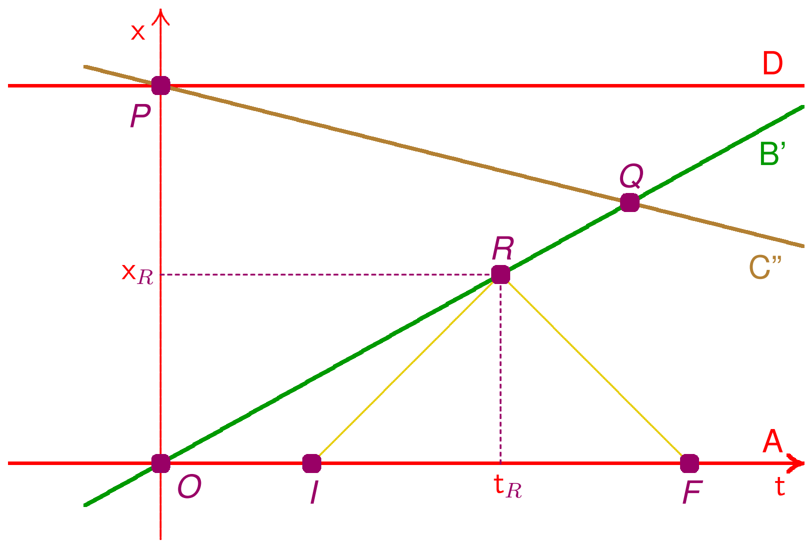

Figure 1.1: *Example of a Minkowski diagram showing the world lines of 4 observers and of a light pulse emitted by one of the observers (A), reflected by a second observer (B') and received back by the first one. The world lines of observers are labeled by capital letters in normal font and the events by italic capital letters in purple color. The world lines of light pulses are drawn in yellow. Also indicated in the diagram are the coordinates of the light-pulse reflection event in the adopted referential, which is that associated to observer A.*

Figure 1.1 presents an illustration of the concepts introduced above. The world lines of four observers named A, D, B' e C" are shown. The coordinate axes indicate time and positions as determined by observer A, which means that, in the adopted referential, observer A is at rest, his position does not change and, consequently, his world line is a horizontal straight line in the graph. The same is true of observer D, who does not move with respect to A but is simply located elsewhere. Observer B' goes by observer A in the event denoted *O* and, from then on, she moves away from A. Some time after the passage of B', observer A sends a light pulse which is reflected by B' and received back by A. This pulse's events of emission, reflection and reception are denoted by *I*, *R* and *F*, respectively. The fourth observer, C", crosses observer D at the event denoted *P* and, from then on, moves away from D but towards A e B', passing by the latter at event *Q*.

It seems worthwhile to summarize the conventions adopted in Figure 1.1, which will be used systematically in this work. Events are indicated by capital letters in *italic* and in *purple color*. The color used to represent world lines of light pulses is yellow. To each observer is attributed a capital letter from the beginning of the alphabet in **regular font**, and a color. Two observers that are at rest with respect to each other receive the same color, but two observers in relative motion receive different colors. As a convenience for the reader who has difficulty distinguishing colors, or wishes to print the text in black and white, an additional convention, redundant with color, is also used: referentials of observers in relative motion are distinguished by the number of "primes" accompanying the associated symbols. In the above example, the symbols A and D, without prime on either of them, were used, since the corresponding observers are at rest with respect to each other, but B', not B, was used to indicate that the observer B' is in motion with respect to observers A e D. Similarly, C" was used to indicate that this observer is moving, not only with respect to A and D, but also with respect to B'. Such a convention shall be particularly useful to distinguish the coordinates attributed to the same event by two observers in relative motion. As an example, the time and space coordinates attributed by observer A to the light-pulse reflection event *R* would be denoted t_R and x_R. Since, in the diagram of Figure 1.1, the coordinate system is that used by observer A, these coordinates can be easily obtained by orthogonal projections on the axes, as can be seen in the figure. In contrast, the coordinates attributed by B' to the same event, which would be denoted t'_R and x'_R, cannot be easily visualized in the graph although, in this case, being *R* an event of the life of B', if, as would be natural, this observer chooses his own position as origin of his spatial coordinate system, one will have $\mathrm{x}'_R = 0$, evidently.

Chapter 2

Introduction and Principles

2.1 Introduction

In this chapter, a few important aspects of classical physics are briefly reviewed, more specifically those related to the motion of bodies and the propagation of waves. Then the two fundamental postulates of Relativity are enunciated, with emphasis upon their relations to and differences from the classical concepts previously recalled.

2.2 Classical Mechanics

Classical or Newtonian mechanics sets forth the general laws of motion of bodies when their velocities are much smaller than the velocity of light.

Newton's first law states that there exists a class of referentials in which all bodies, free from any influence of other bodies, are in uniform rectilinear motion or at rest. Such referentials are called ***inertial referentials***. A referential in uniform rectilinear motion with respect to an inertial referential is also inertial.

In Newtonian mechanics, ***time is absolute***, that is, it is the same in all referentials. ***The laws of classical mechanics are formulated in inertial referentials and are the same in all inertial referentials***.

Newton's second law introduces the concept of ***force*** to represent the influence of other bodies on the motion of a given body. It states that ***force produces acceleration***.

Newtonian mechanics embodies ***Galileo's principle*** according to which it is not possible, through the observation of physical phenomena happening within a closed laboratory, to determine whether the latter is in uniform rectilinear motion or at rest.

In the conceptual framework of Newtonian mechanics, there is no limit to the value that the propagation velocity of a particle or signal can assume. As a consequence, that theory admits the notion of instantaneous influence at a distance. As emphasized below, in Einstein's Relativity, a more general and precise theory, there is a limit to the propagation velocity of any signal and instantaneous action at a distance can only be assumed as an approximation valid in certain circumstances.

2.3 Waves in a material medium

The laws of propagation of material waves are normally ***formulated in a referential in which the propagation medium is at rest***. For a sound wave, for example, the propagation velocity with respect to the medium is a characteristic property of that medium.

If the material medium of propagation is moving in the referential of the observer, the observed wave propagation velocity results from the combination of the medium velocity with the wave propagation velocity with respect to the medium. Therefore, the velocity attributed to the wave by the observer depends on the velocity of the medium.

The observed frequency is affected by the velocity, with respect to the propagation medium, of the emitter as well as of the receiver.

2.4 Light

Light is a wave phenomenon, more precisely, an electromagnetic wave which, contrary to sound, can propagate in regions of space in which there is no matter. All attempts to identify a propagation medium (the hypothetical ***ether***) have failed and it may be stated that light propagates in vacuum.

2.5 Postulates of Special Relativity

Special Relativity is essentially contained in two postulates that can be enunciated very simply:

- ***All laws of Physics are valid in all inertial referentials.***
- ***The velocity of light in vacuum is a universal constant***, independent of frequency and of the source's motion.

However, the simultaneous acceptation of these postulates immediately leads to a dramatic conclusion: ***the velocity of light assumes the same value in all inertial referentials***. This conclusion conflicts with our intuitive notions on the combination of velocities but its validity was confirmed by a famous experiment realized by ***Michelson and Morley***, who showed that the velocity of light in a terrestrial laboratory is not affected by the Earth's orbital motion around the Sun. Given this, the development of Special Relativity will clearly require ***a profound revision of*** our ***concepts of space and time***.

Simply on the basis of the two postulates stated above, the possibility of some particle or signal propagating faster than light could not be discarded. Such entities, known in the literature as ***tachyons***, have been a frequent subject of theoretical study and experimental search. It may be claimed, however, that there is no convincing evidence for their existence, which would be quite problematic from a conceptual point of view for it would imply the possibility of the future influencing the past. This question will not be further addressed in this work, which shall adopt the additional hypothesis that the speed of light is a limit that is reached by electromagnetic waves and possibly other fields or particles, but is never overcome.

Chapter 3

Bondi's k factor

3.1 Introduction

In the development of Special Relativity, an essential task is to establish the relationships between measurements of time, distance and other related quantities, performed by two inertial observers. For this purpose, it is first necessary to characterize the motion of one of the observers with respect to the other. The procedure most usually used is to stipulate the relative velocity of the observers. However, as already anticipated in the previous chapter, the concept of velocity itself will need to be reassessed in Special Relativity and to invoke it from the start is questionable from a conceptual point of view and, in fact, fairly inconvenient in practice.

Seeking an alternative approach, the cosmologist Hermann Bondi proposed as a starting point to consider light pulses emitted by the first observer and received by the second. The ratio between the reception and emission time intervals of such pulses, which Bondi called k ***fator***, can be conveniently adopted as the basic quantity to specify the relationship between the observers. If these are at rest with respect to each other, the k factor will obviously be equal to one, but if they are moving towards or away from each other, the value of k will be different from one. One only needs to imagine a sequence of pulses emitted (and therefore also received) at regular time intervals to be able to interpret the k factor as the ration between the repetition periods of reception and emission. The fact that this ratio differs from unity when the receiver is moving with respect to the emitter is nothing else than the ***Doppler effect***, well known from classical wave theory.

Bondi's k factor will be used systematically in all developments of this work as basic quantity to characterize the relation between observers in uniform rectilinear motion with respect to each other. In the present chapter, a few elementary properties of the k factor will be discussed on the basis of the Special Relativity postulates

3.2 Situation

Consider three observers A, B' and C, such that C is distant from A but at rest relative to A and such that B' is moving away from A and towards C. Each observer carries a clock and observers A e B' also carry devices able to emit light pulses at regular intervals.

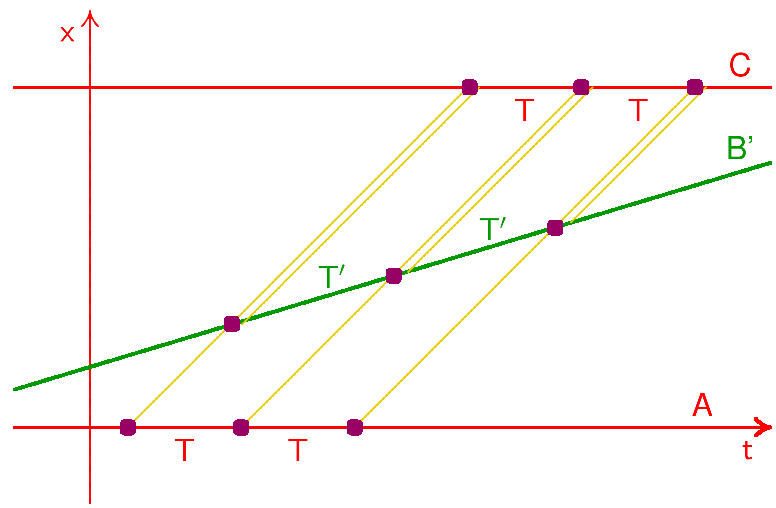

Figure 3.1: *Minkowski diagram showing the situation considered to introduce and discuss Bondi's factor. Observers A and C are distant but at rest with respect to each other. Observer B' is moving away from A and towards C. The world lines of these three observers are represented in the coordinate system associated to A's referential. The world lines of light pulses emitted by A and B' are also drawn. The events of emission and detection of light pulses are represented by dots in purple color.*

Observer A emits light pulses separated by equal time intervals T (as measured by A's clock)

These pulses are received by B' at regular time intervals T' (as measured by the clock carried by B'). Every time observer B' receives a light pulse coming from A, he also emits a light pulse.

Observer C receives the pulses emitted by A and B'.

This situation may be visualized on the Minkowski diagram of Figure 3.1.

3.3 Definition

Because she is moving with respect to A, observer B' receives the pulses emitted by A at intervals T', different from T. This effect, known as Doppler effect, is quite familiar. For example, the sound of an airplane has higher pitch when the plane is flying towards the person hearing the noise, and has lower pitch when the plane is flying away. The motion of the source towards the receiver results in an increase of reception frequency and, therefore, a decrease in reception period, whereas the motion of the source away from the receiver results in a decrease of reception frequency and, therefore, an increase in reception period. Intuitively, if the emitter is moving away from the receiver (or vice versa), each wave crest must travel a larger distance than the previous one, resulting in an increase in the reception interval of successive crests. The same argument applies to the pulses considered in the above situation: since observer B' is moving away from A, each pulse emitted by A must travel a distance larger than the previous one to reach B'. Therefore, the interval of reception of the pulses by B' is larger that the interval of emission of these pulses by A. The expression ***Bondi's* k *factor***, or simply k ***factor***, will be used for the ratio between the reception interval T' of the pulses by B' (measured by the clock of B') and the emission interval T of the pulses by A (measured by the clock of A):

$$\boxed{k = \frac{\text{reception interval by B' (clock of B')}}{\text{emission interval by A (clock of A)}} = \frac{T'}{T}\,.} \tag{3.1}$$

This is Bondi's factor characterizing the motion of B' with respect to A. The roles of A and B' can clearly be swapped, so that the same k factor characterizes the motion of A with respect to B'.[1] In the present case, $k > 1$, for the observers are moving away from each other.

3.4 Analysis

Given the above definition of the k factor, the reception interval of pulses by B' is $T' = kT$. Therefore, the emission time interval of pulses by B' is $T' = kT$ also. Since, by the second postulate, all light pulses travel at the same speed, the pulses emitted by A and B' travel abreast (in pairs) toward C and are received at the same interval. This interval, measured by the clock of C, is equal to the interval T of emission of pulses by A, since C is at rest with respect to A.

[1] This statement is valid in the case of light because the velocity of light is the same for all observers. In the case of a sound wave, it would be necessary to consider, not only the relative motion of the source and the receiver, but also the motions of the source and the receiver with respect to the propagation medium.

3.5 Summary

A		B'		C
emission interval T	$\longrightarrow$	reception interval kT		
		emission interval kT	$\longrightarrow$	reception interval T

3.6 Conclusion

The ratio between the reception interval of pulses by C (measured by the clock of C) and the emission interval by B' (measured by the clock of B') is

$$\boxed{\bar{k} = \frac{\text{reception interval by C (clock of C)}}{\text{emission interval by B' (clock of B')}} = \frac{T}{T'} = \frac{T}{kT} = \frac{1}{k}\,.} \tag{3.2}$$

The $\bar{k}$ factor thus defined characterizes the motion of observer C with respect to observer B' or, equivalently, the motion of observer B' with respect to observer C. In this case, $\bar{k} < 1$, since the two observers in question are moving toward each other.

3.7 Synopsis

Relative motion of emitter and receiver	$\dfrac{\textit{reception interval}}{\textit{emission interval}}$
away from each other	k
toward each other	$1/k$

3.8 Comments

The notion of a k factor different from unity is not peculiar to Special Relativity, it is merely the Doppler effect. But the relationship displayed in the above synopsis ***is*** characteristic of Special Relativity, for it embodies the independence of the velocity of a light signal from the velocities of the source and of the receptor.

The k factor is quite convenient to characterize the relative motion of two inertial observers but, once this concept is introduced, the relative velocity v, more customarily utilized as starting point in presentations of Special Relativity, can naturally be deduced as a function of k. This will be done in the next chapter.

Chapter 4

Relation between the k factor and the relative velocity v

4.1 Introduction

Most presentations of Relativity utilize the relative velocity to specify the kinematic relation between two observers. Although less convenient than Bondi's factor, this variable is important and this chapter is devoted to establishing the relationship between the two quantities.

4.2 Situation

An observer B' is in uniform rectilinear motion with respect to another observer A. Observer A carries a device capable of emitting and detecting light pulses, whereas observer B' carries a mirror capable of reflecting the pulses sent by A.

When B' passes by A, both reset their clocks to zero; denoting this event by *O*, one has therefore $\mathrm{t}_O = \mathrm{t'}_O = 0$.

Also when B' passes by A, the latter sends a first light pulse to B'. Since A and B' are at the same place at that instant, this pulse is reflected essentially instantaneously by the mirror and the reflected pulse is received by A at the same instant. That is, one may consider that the events of emission, reflection and reception of this first pulse all coincide with event *O*.

Observer B' then begins to move away from A. This motion of one observer with respect to the other may be characterized by the Bondi factor k.

Some time later, observer A sends a second light pulse to B'; this pulse is reflected by the mirror and received back by A. One may denote by *I*, *R* and *F*, respectively, the events of emission (by A), reflection (by B') and reception (by A) of this pulse.

The situation described above is represented in the Minkowski diagram of Figure 4.1. Naturally, the first pulse, the existence of which has arbitrarily short duration, is not visible. Besides the elements introduced above, the graph shows the necessary construction in order to extract the coordinates $(\mathrm{t}_R, \mathrm{x}_R)$ of the event *R* of reflection of the second pulse; these coordinates play a central role in the analysis of the next section.

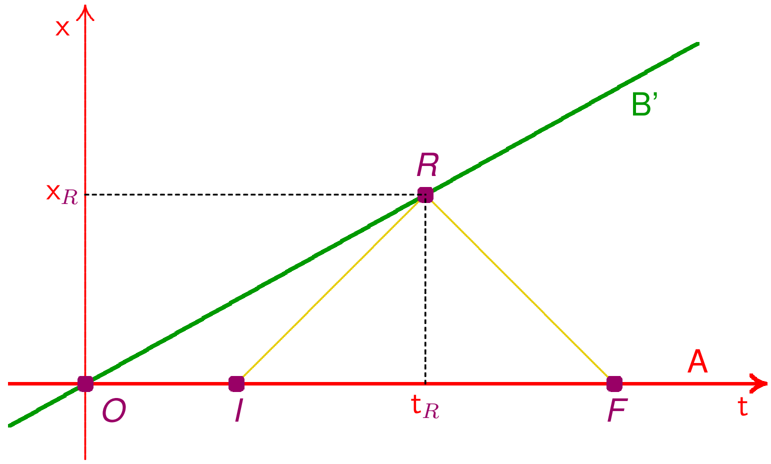

Figure 4.1: *Minkowski diagram showing the situation considered in order to establish the relation between the Bondi factor and the relative velocity of a pair of observers. The graph displays the world lines of observers A and B' and of the (second) light pulse sent by the first observer and reflected by the second. The graphical determination of the coordinates of the reflection event, in the referential of A, is also made explicit.*

4.3 Analysis

In the notation introduced at the beginning of this text, the emission times of the two pulses, measured by the clock of A, are t_O and t_I, and the emission interval of the pulses is $t_I - t_O$. The reflection times of these pulses, measured by the clock of B', are t'_O and t'_R. Naturally, a reflection may be considered as a reception immediately followed by a reemission; therefore the reception interval of the pulses by observer B' is $t'_R - t'_O$. By the definition of the k factor, one may then write

$$t'_R - t'_O = k(t_I - t_O) \ , \tag{4.1}$$

or, given that $t_O = t'_O = 0$,

$$t_I = \frac{t'_R}{k} \ . \tag{4.2}$$

In the same notation, the reception times, by observer A, of the reflected pulses, are t_O and t_F, respectively, and the reception interval of these pulses is $t_F - t_O$. As was already pointed out, the reflection times of the pulses by B' may be interpreted as emission times of the reflected pulses and, therefore, the emission interval of these reflected pulses, measured by the clock of B', is $t'_R - t'_O$. The definition of the k factor may be invoked again in order to relate the reception interval, by A, of the reflected pulses, to the emission interval, by B', of these pulses. The relationship in question is, clearly,

$$t_F - t_O = k(t'_R - t'_O) \ , \tag{4.3}$$

or simply,

$$t_F = k\, t'_R \ . \tag{4.4}$$

Having registered the emission time t_I of the second pulse and the reception time t_F of the corresponding reflected pulse, observer A may attribute a time and a position to the reflection event *R*. Since the second pulse propagated with the constant velocity c, it took the same time to reach event *R* and to return. Therefore, the time at which it was at *R* is the arithmetic mean between the emission time and the reception time. That is, observer A attributes to event *R* the time

$$t_R = \frac{t_I + t_F}{2} = \frac{t'_R/k + k\,t'_R}{2} = \frac{k^2+1}{2k} t'_R \,, \tag{4.5}$$

where use was made of relations (4.2) and (4.4).

The distance between event *R* and observer A, which is the position coordinate x_R attributed by A to event *R*, is half the total distance traveled by the second pulse from its emission to its reception, after reflection by B'. Since the pulse propagates with velocity c, this yields, invoking again relations (4.2) and (4.4):

$$x_R = \frac{c(t_F - t_I)}{2} = \frac{c(k\,t'_R - t'_R/k)}{2} = \frac{c(k^2-1)}{2k} t'_R \,. \tag{4.6}$$

4.4 Conclusion

Since observer B' was present at event *O* and at event *R*, for observer A, observer B' traveled a distance x_R during the time t_R. From this, the velocity v of B' with respect to A may be calculated simply as

$$v = \frac{x_R}{t_R} \,. \tag{4.7}$$

From results (4.5) and (4.6), one obtains, after some elementary simplifications,

$$\boxed{v = c\frac{k^2-1}{k^2+1} \,.} \tag{4.8}$$

This is the expression of the relative velocity of two observers whose motion away from each other is characterized by the Bondi factor k. It can be readily verified that $k = 1$ corresponds to $v = 0$ and that $v < c$ always.

In order to derive the inverse relation, one only needs to rewrite (4.8) in the form

$$k^2(c - v) = c + v \tag{4.9}$$

to obtain

$$\boxed{k = \sqrt{\frac{c+v}{c-v}} \,.} \tag{4.10}$$

4.5 Comments

The relationship between the Bondi factor k and the relative velocity v obtained in the previous section is valid in the case of two observers who are moving away from each other.

As shown in the previous chapter, if the observers were moving towards each other, the Bondi factor would be

$$\bar{k} = \frac{1}{k} = \sqrt{\frac{c - v}{c + v}} . \tag{4.11}$$

Expressions (4.10) e (4.11) are encountered in the majority of Relativity textbooks in the discussion of the Doppler effect[1] for light (or for an electromagnetic wave in general), for the situations in which the detector moves away or towards the source, respectively.

It may be worth while to emphasize that the quantity v considered here is, to be precise, the modulus or absolute value of the relative velocity of the observers. A more conventional approach consists in attributing arbitrarily a positive sense to the line of motion of the second observer with respect to the first and in considering $v > 0$ if the sense of motion coincides with this arbitrary sense and $v < 0$ if they are opposite. Such an attribution specifies the relative motion by the value of $v \in]-c, +c[$, for the whole duration of the motion, without distinguishing between the situations in which the observers are moving away or toward each other. In contrast, in Bondi's approach, which is used in this text, if the relative motion is specified by $k > 1$ in the phase of the motion when the observers are moving away from each other, the same motion had to be specified by $\bar{k} = 1/k < 1$ in the phase of approximation of the observers.

[1] In the form presented here, these expressions give the ratio between the periods of reception and emission. If one considers frequencies rather than periods, the expressions are exchanged, since frequency is the inverse of period.

Chapter 5

Time dilation

5.1 Introduction

One frequently hears the affirmation that Einstein established that time is relative. A more precise statement would be that observers in relative motion will attribute different values to the interval of time between two events. For example, if an observer compares to his own clock the clock carried by an other observer in motion with respect to him, he will conclude that the moving clock is running slow, for the interval between ticks of the moving clock is larger than the interval between ticks of his own clock. The ratio between these intervals is a simple function of the Bondi factor - or the velocity - which characterizes the motion of one observer with respect to the other.

5.2 Situation

The situation to be considered is the same as in the previous chapter and is represented in the Minkowski diagram of Figure 4.1. The pair of events of interest are two ticks of the clock carried by observer B'. The first tick occurs at event *O*, when the observers pass by each other. The second tick occurs when the second light pulse is reflected by the mirror carried by B', that is, at event *R*.

As in the previous chapter, it is convenient to suppose that the observers both reset their clocks to zero when they pass by each other, so that the clocks both indicate zero time when the first tick occurs: $\mathrm{t}_O = \mathrm{t'}_O = 0$. The time indicated by the clock of B' when the second tick occurs is denoted by $\mathrm{t'}_R$ and the time attributed by observer A to this event is denoted by t_R. The interval between the ticks is therefore $\mathrm{t'}_R$ by the clock of B' and t_R by the clock of A. The ratio $\mathrm{t}_R/\mathrm{t'}_R$ between these intervals is the quantity of interest. If time were universal, flowing equally for all observers, this ratio would be, evidently, equal to unity.

5.3 Analysis

Since the second tick of the clock carried by observer B' coincides with event *R*, the time indicated by the clock of observer A when it happens is the time attributed by A to event *R*, which was already calculated in the previous chapter, with the result [see expression (4.5)]

$$t_R = \frac{t'_R/k + k\,t'_R}{2} \ . \tag{5.1}$$

The ratio between the intervals separating the ticks, as measured by each observer, is therefore

$$\frac{t_R}{t'_R} = \frac{1/k + k}{2} \equiv \gamma \ , \tag{5.2}$$

where the conventional notation γ was introduced for this important quantity, known as the ***Lorentz factor***. Using expression (4.10) previously obtained for the k factor in terms of the relative velocity v, it is easy to obtain the expression of the Lorentz factor in terms of v:

$$\gamma = \frac{1}{2}\left(\sqrt{\frac{c-v}{c+v}} + \sqrt{\frac{c+v}{c-v}}\right) = \frac{c}{\sqrt{c^2 - v^2}} \ , \tag{5.3}$$

or

$$\boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \ .} \tag{5.4}$$

Naturally, $\gamma = 1$ if the observers are at rest with respect to each other. For observers in relative motion, $\gamma > 1$ always and the γ factor grows indefinitely when the relative velocity of the observers approaches the velocity of light.

5.4 Summary

In order to express in a more general notation the result obtained above, it is convenient to adopt the Greek letter Δ to indicate an interval of variation of some quantity. For example, the notation $\Delta t'$ indicates the variation (in this case, between events *O* and *R*) of the time indicated by the clock carried by observer B' and the expression Δt refers to the variation of the time marked by the clock of observer A (between the same pair of events). In the development above, one has $\Delta t' = t'_R$ and $\Delta t = t_R$. Therefore, relation (5.2) takes the form

$$\Delta t = \gamma \Delta t' \ . \tag{5.5}$$

The essential difference between observers B' and A is that, for the first, the clock whose ticks are the events of interest is at rest whereas, for the second, that clock is moving with velocity v. In order to emphasize this essential point, it is convenient to use the notation Δt_0 for the time interval indicated by the clock at rest and the notation Δt_v for the corresponding interval, measured by the clock of an observer who sees the first clock moving with velocity v. In the situation analyzed above, one has then $\Delta t' = \Delta t_0$ and $\Delta t = \Delta t_v$, so that relation (5.5) reads

$$\boxed{\Delta t_v = \gamma \Delta t_0 \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1 \ .} \tag{5.6}$$

The phrasing ***proper time interval*** is frequently used in reference to Δt_0, the time interval between ticks of a clock, measured in a referential in which this clock is at rest.

5.5 Conclusion

Watching the pointer's rotation of a moving clock, an observer notes that it indicates a time interval Δt_0. Comparing with the pointer's rotation of the clock at his wrist, he notices that it does not correspond to the same time interval, but to interval Δt_v, such that $\Delta t_v = \gamma \Delta t_0 > \Delta t_0$. Trusting, evidently, his own clock, he concludes that ***the moving clock is running slow***. This phenomenon is known as ***time dilation***.

It is worth while to emphasize the following points:

- It can be easily verified that expression (5.2) is invariant under the exchange $k \leftrightarrow 1/k$ and, therefore, for a given value of the relative velocity, time dilation is the same when the two observers are moving toward each other as when they are moving apart. This difference between the Doppler effect and time dilation should be stressed: in the Doppler effect, one observes an increase in the period when the source is moving away and a decrease in the period when the source is getting closer; in contrast, the passage of time is subject to dilation only, time compression never occurs.
- The effect is reciprocal. For B', it is the clock carried by A which is moving and is, therefore, running slow.
- From a theoretical point of view, time dilation is an irrefutable consequence of the postulates of Relativity.

5.6 Illustration

Time dilation is verified experimentally, for example in the decay times of unstable particles produced by cosmic rays when they impinge on the Earth's atmosphere. One can imagine that such particles possess an "internal clock" which determines their proper lifetimes Δt_0. In the referential of the Earth, the particles propagate with velocity v close (although inferior) to the speed of light. The time they take to traverse the atmosphere is $\Delta t = H/v$, where H is the atmosphere's height, measured in the referential of the Earth.[1] It turns out that $\Delta t > \Delta t_0$, so that, if time dilation did not occur, the particles would vanish before they could be detected in laboratories installed on the Earth's surface. But thanks to time dilation, the lifetime of these particles in the Earth's referential is Δt_v, much larger than Δt_0, since the Lorentz factor γ is much larger than unity. For a particle with velocity sufficiently close to the speed of light, Δt_v will be larger than Δt and the particle will survive long enough to cross the whole atmosphere and be detected at ground level.

[1] The importance of stating explicitely in what referential a height, or a length, is measured, is the topic discussed in the next chapter.

Chapter 6

Length contraction

6.1 Introduction

In order to establish the geometry of space, it is necessary to give meaning to the distance between two points or, more concretely, to the length of a rigid object. If the object is at rest, there is no difficulty. For example, if an observer is on a platform which is at rest, he will be able to determine its length by walking from an extremity to the other and counting the number of steps. Since the platform is not moving, there is no need to worry about the simultaneity of the events used in the measurement, the observer may take all the time he wants to complete the walk. Naturally, for a more precise determination, use can be made of the fundamental principles of physics and, in particular, of the invariance of the speed of light. One may set a mirror at one of the platform's extremities and, from the other extremity, send a light pulse to be reflected by the mirror. Letting *E* be the pulse's emission event and *D* the event of detection of the reflected pulse, the length of the platform will be given by $L_0 = c\frac{t_D - t_E}{2}$, where c is the speed of light and the times t_D and t_E are measured by the clock of the observer who is at rest on the platform. The length thus determined is known as ***rest length***, or ***proper length*** of the platform, since it is measured by an observer who is at rest relative to the platform.

When the object whose length is to be measured is moving with respect to the observer who will perform the measurement, care must be taken to guarantee that the events selected at each extremity of the object, in order to calculate the distance between them, occur ***simultaneously*** for the observer in question.

6.2 Situation

Observer A wishes to determine the length L (for him) of a platform which is moving with velocity v, corresponding to the Bondi factor k. Two observers B' and C' are at rest on the platform, each at an extremity. Both carry a mirror. For them, the platform's length is L'. Since the platform is at rest with respect to these observers, this is the proper length L_0 of the platform. One has, then, $L' = L_0$. One may assume that this length was previously measured by B', for example, by sending a light pulse to the mirror of C' and detecting the reflected pulse, following the procedure described in Section 6.1. One may further assume that B' communicated the result to A.

In order to determine the length of the platform, as observed by him, A needs to calculate the distance between two events occurring simultaneously (for him) at each extremity of the platform. These events shall be reflections of light pulses by the mirrors.

It shall be assumed that B' passes by A before C' and, in order to simplify the arguments, it shall be further assumed that, when C' passes by A, both reset their clocks to zero. One may imagine that A sends a pulse to C' at that event; this pulse (which shall be referred to as pulse 0) is received and reflected instantaneously by C', and the reflected pulse is received instantaneously by A. In other words, the events of emission, reflection and reception of this pulse 0 all coincide and this event will be denoted by *O*. By notational convention, t_O shall represent the time attributed by observer A to this event and t'_O shall represent the time attributed by observer C' to the same event. In the conditions stated above, $t_O = t'_O = 0$. Evidently, the clock carried by B' should also be reset so that, for B' and C', their clocks will always indicate the same time, since they are at rest with respect to each other.

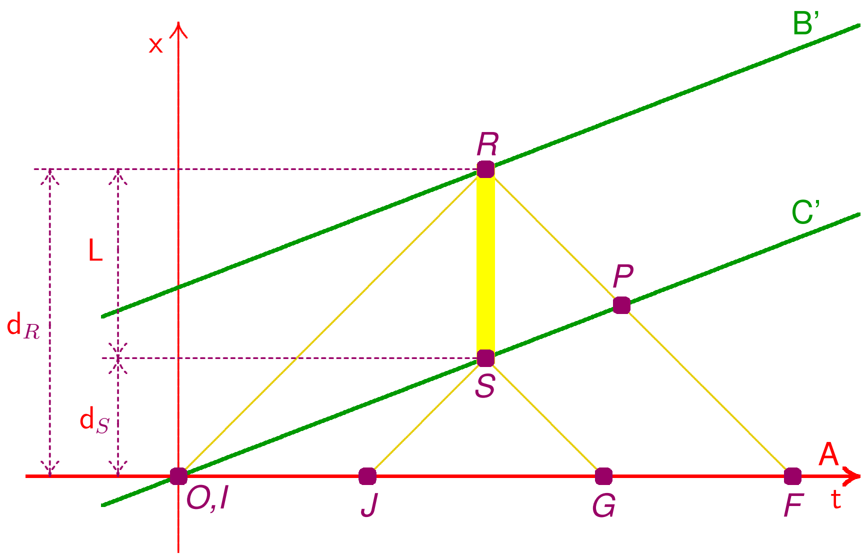

Figure 6.1: *Minkowski diagram showing the situation considered for the analysis of the length contraction of a platform. The diagram's axes refer to the referential of observer A for whom the platform is in motion. The graph shows the world lines of two observers B' e C' located at the platform's extremities. For A, the two light pulses he sends are reflected simultaneously by the mirrors carried by B' and C'. The graphical determination of the length attributed to the platform by observer A is made explicit.*

In order to determine the platform's length, A sends two pulses 1 and 2, which are reflected by B' and C' respectively. Since B' is further away from A than C', the pulse intended to reach B' should be sent first (pulse 1); the pulse that will be reflected by C' is sent at a later time (pulse 2). As shall be seen, the emission times of these pulses shall need to be chosen in such a way that the reflections happen simultaneously for A. In order to simplify the discussion as much as possible, it will be convenient to suppose that the emission of pulse 1 takes place when observer C' passes by observer A. This event will be denoted by *I*, where the letter I serves to recall that this is the initial event in the story of that pulse. With this convention, event *I* coincides, in fact, with event *O* and one has $t_I = t'_I = 0$. The event of reflection of pulse 1 by B' will be denoted by *R* and the event of reception by A of the corresponding reflected pulse will be denoted by *F*, where the letter F reminds the reader of the fact that this is the final event in this pulse's history.

Pulse 2, which will be reflected by C', is sent by A some time later. In analogy with the event succession $I \to R \to F$ in the history of pulse 1, the sequence $J \to S \to G$ will refer to the important events in the history of pulse 2, that is, *J* shall denote the event of emission of this pulse and *G* the event of reception by A of the corresponding reflected pulse. The letter *S* will be used to indicate the event of reflection of this pulse by observer C'.

The Minkowski diagram illustrating the situation described above is presented in Figure 6.1.

6.3 Analysis

The events of emission, by observer A, of pulses 0 and 2 are *O* and *J*, respectively. Therefore, the emission interval of these pulses, measured by the clock of A, is $t_J - t_O$. The reception events of these pulses by observer C' are *O* and *S*, respectively. Therefore, the reception interval of these pulses, measured by the clock of C', is $t'_S - t'_O$. By definition of the Bondi factor

$$t'_S - t'_O = k(t_J - t_O) , \tag{6.1}$$

or simply, given that $t_O = t'_O = 0$,

$$t'_S = k\, t_J . \tag{6.2}$$

Observer C' sends the reflected pulse 0 at event *O* and the reflected pulse 2 at event *S*. Therefore, the emission time interval of these reflected pulses is $t'_S - t'_O$. The reception events, by A, of these reflected pulses are *O* and *G*, respectively. The reception interval of these reflected pulses is therefore, $t_G - t_O$. Invoking again the definition of the Bondi factor, one may write

$$t_G - t_O = k(t'_S - t'_O) , \tag{6.3}$$

or, recalling once more that $t_O = t'_O = 0$,

$$t_G = k\, t'_S = k^2\, t_J , \tag{6.4}$$

where use was made of relation (6.2).

One may now focus the attention on pulse 1. After reflection by B', it passes by C' on its way back to A, an event one may denote by *P*. The detection, by C', of the pulse that passes by him may be considered as a reception followed by immediate reemission. Therefore, one may consider that observer C' emitted pulses at the events *O* (the pulse 0) and *P* (the pulse 1 detected on its way back to A). The emission interval of these pulses is $t'_P - t'_O$. These pulses are received by A at events *O* and *F*, respectively. The reception interval, measured by the clock of the receiver, is then $t_F - t_O$. Using again the definition of the Bondi factor, one may write

$$t_F - t_O = k(t'_P - t'_O) , \tag{6.5}$$

or simply,

$$t_F = k\, t'_P . \tag{6.6}$$

Between events *I* and *P*, pulse 1 went from C' to B' and back to C'. From the point of view of C' and B', it traveled a distance $2\,L'$ between instants t'_I and t'_P; therefore, one has

$$2\,L' = c(t'_P - t'_I) = c\,t'_P\ , \tag{6.7}$$

since $t'_I = 0$. From (6.6) and (6.7), one gets

$$t_F = \frac{2\,k\,L'}{c}\ . \tag{6.8}$$

Since light takes the same time to go from A to B' and to come back, observer A attributes to the event of reflection of pulse 1 by B' the time

$$t_R = \frac{t_I + t_F}{2} = \frac{k\,L'}{c}\ , \tag{6.9}$$

where use was made of relation (6.8), remembering that $t_I = 0$. By the same argument, observer A attributes to the event of reflection of pulse 2 by C' the time

$$t_S = \frac{t_J + t_G}{2} = \frac{1 + k^2}{2} t_J\ , \tag{6.10}$$

where relation (6.4) was used.

In order that the above procedure constitute a measurement, by A, of the platform's length, observer A must send the second pulse at the instant such that the events utilized in the measurement, that is, the reflection events *R* and *S* at the two extremities of the platform, be simultaneous (for him, observer A). In other words, he needs to choose the time t_J at which he emits the second pulse in such a way as to fulfill the condition $t_S = t_R$, which, by (6.9) and (6.10), requires

$$t_J = \frac{2\,k\,L'}{(1 + k^2)c}\ . \tag{6.11}$$

By assumption, observer A knows the values of the Bondi factor k which characterizes the motion of the platform relative to himself and the length L', which was measured and communicated by the observers at rest on the platform. Therefore, observer A can perform the calculation (6.11) and send pulse 2 at the right moment.

One may denote by d_R and d_S the distances, with respect to observer A, of the places of occurrence of events *R* and *S*, respectively. Since light always propagates with velocity c, one has, using (6.8) and recalling that $t_I = 0$ by choice:

$$2\,d_R = c(t_F - t_I) = 2\,k\,L'\ . \tag{6.12}$$

Similarly, using (6.4) and (6.11), one has

$$2\,d_S = c(t_G - t_J) = c(k^2 - 1)t_J = \frac{2\,k(k^2 - 1)}{k^2 + 1} L'\ . \tag{6.13}$$

The length attributed to the platform by observer A will then obviously be

$$L = d_R - d_S = (1 - \frac{k^2 - 1}{k^2 + 1})k\,L' = \frac{2k}{k^2 + 1} L'\ . \tag{6.14}$$

Recalling the relationship previously obtained between the Bondi factor k and the velocity v for two observers moving away from each other, namely [equation (4.10)],

$$k = \sqrt{\frac{c+v}{c-v}}\,, \tag{6.15}$$

it can be easily verified that

$$\frac{2k}{k^2+1} = \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma}\,, \tag{6.16}$$

where γ is the (already well known) Lorentz factor.

6.4 Summary

Inserting the result (6.16) in relation (6.14), one concludes that

$$L = \frac{L'}{\gamma}\,. \tag{6.17}$$

The essential difference between observers B' and A is that, for the first, the platform whose length is being measured is at rest, whereas, for the second, it is moving with velocity v. In order to emphasize this essential point, it is convenient to employ the notation L_0 (already introduced above) for the length of the platform at rest and the notation L_v for the corresponding length, measured by an observer who sees the platform moving with velocity v. In the situation analyzed above, one has then $L' = L_0$ and $L = L_v$, so that relation (6.17) reads

$$\boxed{L_v = \frac{L_0}{\gamma} \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} > 1\,.} \tag{6.18}$$

As already mentioned, the expression ***proper length*** is frequently used in reference to L_0, the length of the platform, or of any other object, measured in the referential in which the object in question is at rest.

6.5 Conclusion

Generalizing the above result, it may be stated that ***to someone who observes a moving object, this object appears contracted in the direction of its motion***. This phenomenon is known as ***length contraction*** or ***Lorentz contraction***, in homage to the physicist who first introduced it, although on the basis of arguments significantly different from those pertinent to Einstein's theory of Special Relativity.

The following points deserve to be emphasized:

- For this effect, it is irrelevant whether the moving object is getting closer or more distant, since the γ factor is the same in both situations, for a given value of the relative velocity v.

- The effect is reciprocal: an object at rest with respect to observer A would seem contracted, by the same γ factor, to observers B' and C'.
- It is a real kinematic effect, not an illusion related to the observation of the object with light.

6.6 Illustration

The propagation toward the Earth's surface and the detection of unstable particles produced by cosmic rays entering the atmosphere, that were mentioned at the end of the previous chapter as an illustration of time dilation, may also be analyzed invoking length contraction.

For this purpose, one needs only adopt the point of view of an observer who accompanies a particle propagating, with velocity v, from the atmosphere's upper region to ground level. For consistency with the conventions adopted in the discussion of the previous chapter, this observer shall be called B' and the observer at rest on the Earth's surface shall be called A. Because of length contraction, the observer who rides with the particle sees the ground (and therefore also the detector), initially at a distance $H' = H_v = H_0/\gamma$, where $H_0 \equiv H$ is the atmosphere's height measured by the terrestrial observer A and γ is the Lorentz factor associated with velocity v. Since the ground is approaching with velocity v, the detector in the terrestrial laboratory takes the time interval $\Delta t' = H'/v = H_v/v = H_0/(\gamma v)$ to reach the particle. If velocity v is sufficiently close to the speed of light, the Lorentz factor γ will be very large and the interval $\Delta t'$ will be less than the proper lifetime Δt_0 of the particle. Consequently, the particle will be reached by the detector before it decays and its detection will occur.

Chapter 7

Combination of velocities

7.1 Introduction

According to the familiar understanding of velocity, the combination law for velocities associated to motions in the same direction is simple arithmetic addition. For example, somebody walking at $5\,\mathrm{km/h}$ on a belt rolling at $3\,\mathrm{km/h}$ is moving at the velocity of $8\,\mathrm{km/h}$ with respect to the airport hall. It is easy to perceive that this law must lose its validity when velocities are appreciable fractions of the speed of light, for it could lead to a resulting velocity larger than the speed of light, which is not allowed in Special Relativity.

In this chapter, the relativistic law of velocity combination is deduced, in the special case of parallel velocities. The combination of Bondi factors, which follows almost trivially from the definition, is obtained first. On the basis of this law and of the relation between Bondi factor and velocity, already established in Chapter 4, the desired result is easily obtained.

7.2 Situation

Consider three observers A, B' and C". Observer B' is moving away from observer A with velocity v with respect to A. Observer C" is moving away from B' with velocity v' with respect to B'. These motions occur in the same direction and the same sense.

The question is: what is the velocity of observer C" with respect to observer A?

To permit the analysis based on the use of the Bondi factor, assume that observer A emits light pulses separated by time intervals T (measured by the clock of A). These pulses are detected by B' at intervals T' (measured by the clock of B'). Every time observer B' detects a pulse coming from A, he also emits a light pulse. The pulses emitted by A and B' propagate together toward observer C", who detects them at intervals T'' (measured by the clock of C").

The situation described above may be visualized on the Minkowski diagram of Figure 7.1.

7.3 Analysis

Let k be the Bondi factor associated with velocity v and k' the factor associated with velocity v'.

The pulses emitted by A at intervals T are received by B' at intervals T'. By definition of the Bondi factor:

$$T' = kT. \tag{7.1}$$

The emission interval of the pulses of B' is also T'. These pulses are received by C" at intervals T''. Invoking again the definition of the Bondi factor, one has

$$T'' = k'T'. \tag{7.2}$$

Since the pulses emitted by A accompany the pulses emitted by B', they also are received by C" at intervals T''. Inserting in relação (7.2) the expression (7.1) of the interval measured by B', the following relation between the intervals measured by A and by C" is obtained:

$$T'' = k'kT. \tag{7.3}$$

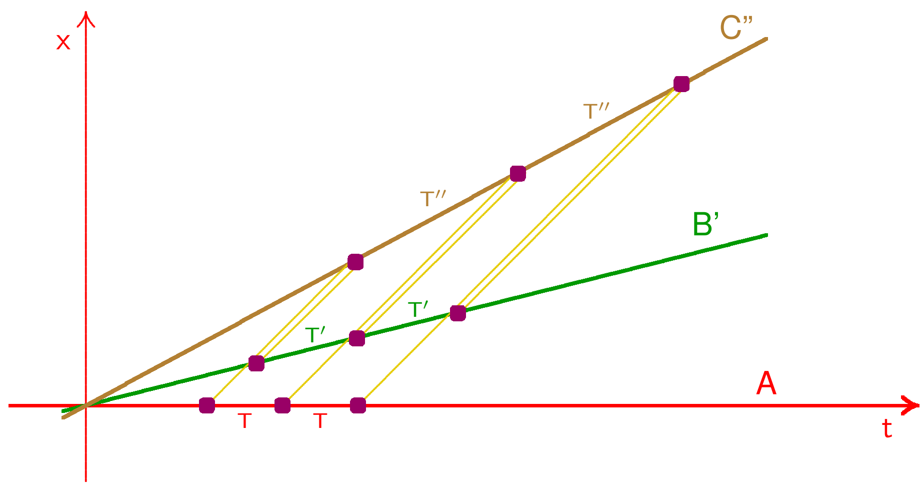

Figure 7.1: *Minkowski diagram showing the situation considered for the analysis of the combination of the Bondi factors associated to motions in the same direction. The diagram's axes refer to the referential of observer A. The world lines of observers A, B' and C", and of the light pulses emitted at regular intervals by observers A and B', are displayed. The intervals of emission and/or reception of the pulses, measured by the observer involved, are indicated.*

Let K be the Bondi factor associated to the motion of C" with respect to A. By the definition of this quantity, one may write the relation

$$T'' = KT. \tag{7.4}$$

Comparing the expressions (7.3) and (7.4) de T'', one concludes that

$$K = k'k\,. \tag{7.5}$$

In words: ***the combination law of Bondi's k factors is simple arithmetic multiplication***.

7.4 Conclusion

Let V be the velocity of observer C” with respect to observer A. Making use of relationship (4.8), one obtains the expression of velocity V in terms of the Bondi factor K:

$$V = c\frac{K^2 - 1}{K^2 + 1} . \tag{7.6}$$

Substituting in this relation the expression (7.5) of K in terms of k' and k and then inserting the expressions of these in terms of v' and v, given by formula (4.10), one has

$$V = c\frac{k'^2 k^2 - 1}{k'^2 k^2 + 1} = c\frac{\frac{c+v'}{c-v'} \times \frac{c+v}{c-v} - 1}{\frac{c+v'}{c-v'} \times \frac{c+v}{c-v} + 1} = c\frac{v'c + vc}{c^2 + v'v} , \tag{7.7}$$

or, finally,

$$V = \frac{v' + v}{1 + \frac{v'v}{c^2}} , \tag{7.8}$$

which constitutes the ***relativistic composition law for collinear velocities***.

7.5 Comments

The above formula makes evident the following properties:

- $V = c$ if $v = c$ or $v' = c$, which corresponds to the invariance of the speed of light: if a body or signal moves with the velocity of light for one observer, it moves with the velocity of light for all observers.[1]

- V is always smaller than c if v and v' are small than c: if a body or signal moves with velocity inferior to the velocity of light for one observer, it moves with a velocity inferior to the velocity of light for all observers.

- If velocities v and v' are both much smaller than the velocity of light c, the second term in the denominator of expression (7.8) will be much smaller than unity and the relativistic combination rule for velocities will be reduced, in very good approximation, to the familiar law of additive combination, $V \simeq v' + v$.

[1]Since v e v' are relative velocities of two observers, they cannot be put exactly equal to c; however, one may consider velocities arbitrarily close to c.

Chapter 8

Twin paradox

8.1 Introduction

In Chapter 5, it was seen that a clock in uniform motion runs slow compared to the clock carried by an inertial observer who witnesses the motion. This result may be applied to the aging of a person, considered to be governed by an internal biological clock. If one imagines a pair of twins born on Earth but who, someday, separate because one of them decides to embark on a long trip to somewhere else in the Universe, one concludes that the twin who remains on Earth must think that his brother is aging more slowly than himself. However, since time dilation is a reciprocal effect, for the traveler, it should be his brother, who remained on Earth, who is keeping younger. As long as the twins are far away from each other, both can be right without manifest conflict since they cannot compare objectively their physical and mental states.

But suppose that, after a few years, the traveling twin decides to return home. When the twins meet again, the comparison will be possible and if one of the brothers is really younger than the other, both will have to agree that this is the case. The present chapter is devoted to the intriguing question.

8.2 Situation

Consider three observers A, B' and C". Observer B' is moving with velocity v with respect to observer A. Observer C" is also moving with velocity v with respect to observer A, in the same direction but ***in the opposite sense***.

Initially, B' and C" are getting closer to A; B' is already near A but C" is still far away. When B' passes by A, both reset their clocks to zero. Denoting by *O* this event, one has then $t_O = t'_O = 0$. At the said event, observer B' sends a first light pulse to A, who receives it essentially instantaneously, since A and B' are at the same spot.

Observer B' begins then to move away from A and toward C". The meeting of observers B' e C" shall be named event *P*. At this event, observer C" adjusts his clock in accordance with that of B', so that $t''_P = t'_P$. As they cross, observers B' and C" both send light pulses to A. These pulses travel together and are received by A at the same event, which will be denoted by *R*.

After his encounter with B', observer C" continues on his way toward A. When C" meets A, he sends a second light pulse to A, who receives it essentially instantaneously. The meeting of C" with A will be named event *Q*.

The sequence of events and motions described above may be visualized on the Minkowski diagram of Figure 8.1.

When observers C" and A are at the same place, they can compare their clocks. The question is: will the clocks of A and C" mark the same time ? Stated in mathematical form, is the equality $t''_Q = t_Q$ satisfied ? Clearly, since B' adjusted his clock by that of A and C" adjusted his clock by that of B', "common sense" would answer ***yes*** to this question.

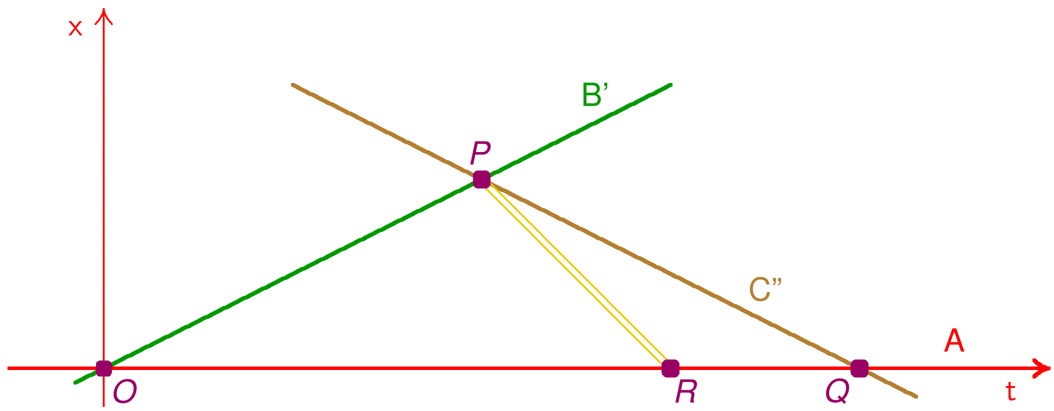

Figure 8.1: *Minkowski diagram showing the situation considered in the analysis of the twin paradox. The diagram's axes refer to the referential of observer A. The world lines of observers A, B' and C" are drawn, as well as those of the light pulses emitted by observers B' and C" when they pass by each other.*

8.3 Analysis

Let t'_P be the time indicated by the clock of B' at the event *P* of his encounter with C". Since C" then adjusts his clock to that of B', the clock of C" marks $t''_P = t'_P$ at that event.

For observer B', the time passed between his encounter with A and his encounter with C" is $t'_P - t'_O = t'_P$, since the clock of B' was adjusted in such a way that $t'_O = 0$. Since observer C" is moving toward A with the same velocity v with which B' moved away from A, he takes the same time (measured by him starting from the meeting with B') to reach A, that is:

$$t''_Q - t''_P = t'_P \ . \tag{8.1}$$

Consequently, at the event *Q* of his encounter with observer A, the clock of C" registers the time

$$t''_Q = t''_P + t'_P = 2\,t'_P \ , \tag{8.2}$$

where it was recalled that C" adjusted his clock in such a manner that $t''_P = t'_P$.

The interval (measured by B') in the emission of the two pulses by B', is $t'_P - t'_O = t'_P$. Let k be the Bondi factor relating A and B'. By the definition of this factor, the interval of reception by A is

$$t_R - t_O = k(t'_P - t'_O) \ . \tag{8.3}$$

Remembering that $t_O = t'_O = 0$, one deduces that, when he receives the second pulse from B', the clock of A indicates the time

$$t_R = k\,t'_P \ . \tag{8.4}$$

Since the first pulse emitted by C" accompanied the second pulse emitted by B', this is also the time registered by the clock of A when he receives the first pulse of C". Given that observer C" is moving toward A with the same velocity with which observer B' was moving away from A, the Bondi factor relating A and C" is $1/k$ (see Chapter 3). The interval of emission of the two pulses by C" is $t''_Q - t''_P$, therefore the interval of reception of these pulses by A is

$$t_Q - t_R = \frac{1}{k}(t''_Q - t''_P) \ . \tag{8.5}$$

Thus, when A receives the second pulse from C", his clock registers the time

$$t_Q = t_R + \frac{1}{k}(t''_Q - t''_P) = k\,t'_P + \frac{1}{k}t'_P \ , \tag{8.6}$$

where use was made of relations (8.4) e (8.1).

Summarizing, the times indicated by the clocks of observers A and C" at the event of their meeting are given by expressions (8.6) and (8.2).

8.4 Conclusion

It can be seen from (8.6) and (8.2) that times t_Q and t''_Q indicated by the clocks of A and C" when the meeting of these two observers happens are not the same. The ratio between them is

$$\frac{t_Q}{t''_Q} = \frac{k + \frac{1}{k}}{2} \ , \tag{8.7}$$

Using expression (4.10) of the k factor in terms of the relative velocity v, one obtains (see the calculation done in the discussion of time dilation, in Chapter 5):

$$\boxed{t_Q = \gamma\, t''_Q \ \ \text{with} \ \ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1 \ .} \tag{8.8}$$

This result is formally identical to that obtained in the discussion of time dilation and may, in fact, be interpreted as a manifestation of this effect.

However, the situation here is significantly different from that considered in Chapter 5. ***The effect here is not reciprocal***. Since A and C" are comparing their clocks ***at the same event*** (the meeting between them), they must agree on the question whether one of the clocks is ahead of the other.

8.5 Illustration

The above conclusion may be invoked to answer the question raised in the introduction to this chapter.

Suppose observer A lives on Earth and is the father of twins John and Peter. Observer B’, an extraterrestrial being traveling in his spaceship, passes by A and picks John up, whereas Peter remains on Earth with A. After a few years of traveling, the spaceship of B’ passes by another spaceship coming in the opposite direction, with the same velocity (with respect to the Earth). In command of that spaceship is observer C”, another extraterrestrial. Longing for home, John transfers himself to the spaceship of C” and, after a few more years of traveling, he is back on Earth and re-encounters his father A and his brother Peter.

When A compares his reunited twins, he notices that John***, who traveled, is now younger than*** Peter***, who stayed home!*** Note that everyone has to agree on this. ***The situation is not symmetric***, for John jumped from one inertial referential to another, whereas Peter always remained in the same inertial referential. As he transfered himself from Earth to the first spaceship, from this to the second spaceship, and from the latter back to Earth, John necessarily ***underwent acceleration***. In contrast, Peter was not submitted to any acceleration (to the extent that Earth may be considered as constituting an inertial referential[1]).

The drama imagined above belongs, for the time being, to the realm of science fiction only. However, the physical effect that is involved is verified in high-precision experiments comparing clocks carried by airplanes to clocks at rest on Earth.

[1]This assumption obviously disregards the Earth’s orbital motion about the Sun.

Appendix A

Examples

A.1 Bondi factor - example

The diagram of Figure A.1 illustrates the situation from the point of view of observer A.

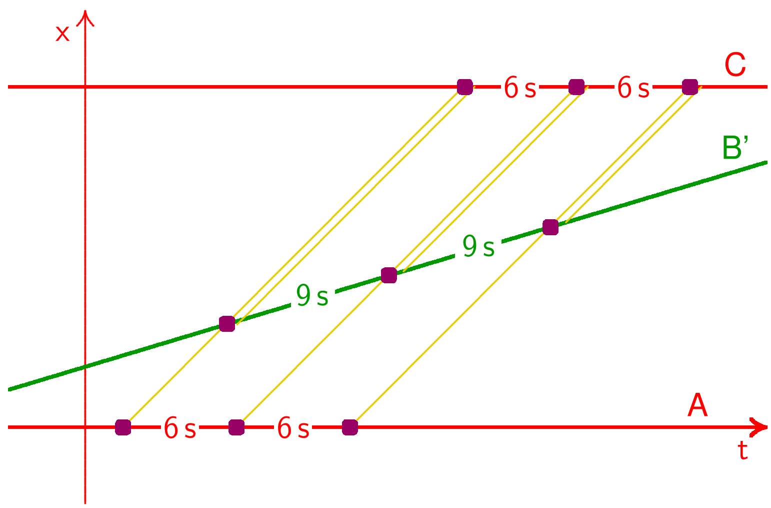

Figure A.1: *Minkowski diagram showing the situation considered to illustrate the discussion of the Bondi factor. Observers A and C are distant from each other but at rest with respect to each other. Observer B' is moving away from A and toward C. The time intervals measured by each observer between the events of emission and detection of light pulses are indicated.*

The light pulses emitted by A at intervals of $6\,\mathrm{s}$ are received by observer B', who is ***moving away*** from A, at intervals of $9\,\mathrm{s}$. Therefore, the ***Bondi factor*** characterizing the relation between A and B', takes the value

$$k = \frac{9\,\mathrm{s}}{6\,\mathrm{s}} = \frac{3}{2}\,. \tag{A.1}$$

Every time B' receives a light pulse coming from A, he also emits a light pulse. The light pulses emitted by A and B' travel abreast to C, who receives them simultaneously. Observer C is at rest with respect to A and observer B' is moving toward C with the same velocity (in absolute value) with which he is moving away from A.

Since C is at rest with respect to A, he receives the pulses of A at intervals equal to the emission intervals, of $6\,\mathrm{s}$. Clearly then, C also receives the pulses emitted by B' at intervals of $6\,\mathrm{s}$. Consequently, for these pulses, the ratio between the reception and the emission intervals is

$$\frac{6\,\mathrm{s}}{9\,\mathrm{s}} = \frac{2}{3} = \frac{1}{k}\,. \tag{A.2}$$

In words: ***the Bondi factor associated to a pair of observers who are moving toward each other with a certain relative velocity is the inverse of the Bondi factor associated to a pair of observers who are moving apart with the same relative velocity (in absolute value)***.

A.2 Calculation of the relative velocity given the Bondi factor - example

The diagram displayed in Figure A.2 illustrates the situation from the point of view of observer A. Let $k = 3/2$ be the Bondi factor relating observers A e B'. When the two observers pass by each other, both reset their clocks to zero and exchange light pulses.

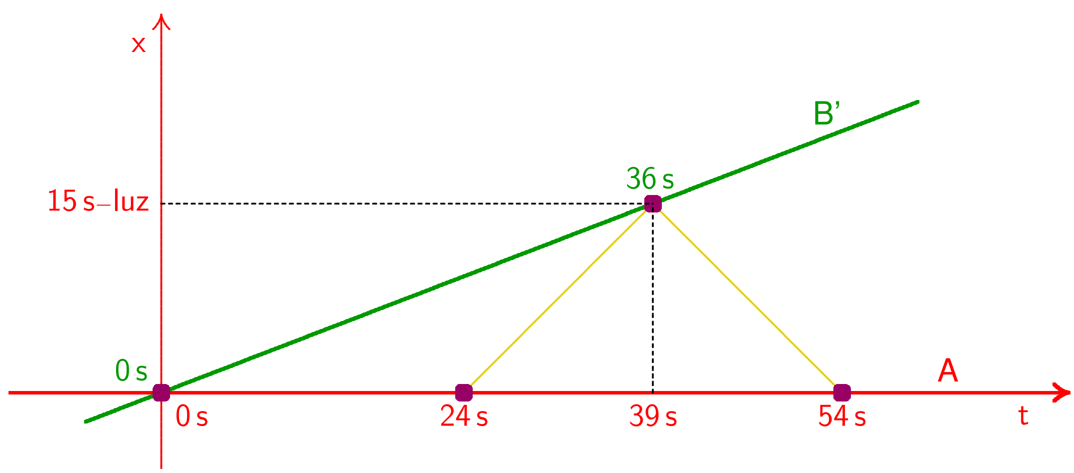

Figure A.2: *Minkowski diagram presenting the situation considered in order to establish the relation between the Bondi factor and the relative velocity of a pair of observers. The world lines of the two observers A and B', and of the (second) light pulse sent by the first observer and reflected by the second, are drawn. The graphic determination of the coordinates of the reflection event, in the referential of A, is made explicit. The values indicated correspond to the case $k = 3/2$.*

Some time later, B' receives a second light pulse coming from A; let us assume that the clock of B' marks $36\,\mathrm{s}$ at that event. The reception interval of the two pulses by B' is, then, $36\,\mathrm{s}$. Therefore, the emission interval of the pulses by A was

$$\frac{36\,\mathrm{s}}{k} = 36\,\mathrm{s} \div \frac{3}{2} = 24\,\mathrm{s}\,. \tag{A.3}$$

Since the first pulse was sent by A when his clock read $0\,\mathrm{s}$, the second pulse was emitted when the clock of A indicated $24\,\mathrm{s}$.

Observer B' sends the pulse back to A. The emission interval between the two pulses sent to A by B' is $36\,\mathrm{s}$, by the clock of B'. The reception interval of these pulses by A is, therefore,

$$36\,\mathrm{s} \times \mathrm{k} = 36\,\mathrm{s} \times \frac{3}{2} = 54\,\mathrm{s}\ . \tag{A.4}$$

Since the first pulse was received by A at the instant $0\,\mathrm{s}$, the clock of A reads $54\,\mathrm{s}$ when he receives the reflected pulse, which is the second pulse coming from B'.

For A, the reflection event of the second pulse occurred at the middle instant between the emission and reception instants, that is, when the clock of A marked

$$\frac{24\,\mathrm{s} + 54\,\mathrm{s}}{2} = 39\,\mathrm{s}\ . \tag{A.5}$$

Consequently, for A, the second light pulse he sent took

$$39\,\mathrm{s} - 24\,\mathrm{s} = 15\,\mathrm{s} \tag{A.6}$$

to reach the event of reflection by B'. Hence, this event happened at a distance of $15\,\mathrm{light{-}s}$. Since B' was present at the reflection event (he provoked it!), A may conclude from this analysis that B' covered $15\,\mathrm{light{-}s}$ in $39\,\mathrm{s}$. The velocity of B' with respect to A is then

$$\mathrm{v} = \frac{15\,\mathrm{light{-}s}}{39\,\mathrm{s}} = \frac{5}{13}\,\mathrm{c}\ . \tag{A.7}$$

The reader may verify that the relation between k and v, in this example, corresponds to the general formula [see relation (4.10)]

$$\mathrm{k} = \sqrt{\frac{\mathrm{c}+\mathrm{v}}{\mathrm{c}-\mathrm{v}}}\ , \tag{A.8}$$

that is, he may check that

$$\sqrt{\frac{1+5/13}{1-5/13}} = \frac{3}{2}\ . \tag{A.9}$$

A.3 Time Dilation - example

In order to illustrate this effect, one may consider the situation presented in section A.2 and visualized in Figure A.2, taking advantage of much of the associated analysis. The Bondi factor relating observers A and B' is $\mathrm{k} = 3/2$.

When the two observers pass by each other, both set their clock to zero; therefore, at this event, the clock of A marks the time $0\,\mathrm{s}$ and the clock of B' also marks $0\,\mathrm{s}$.

However, according to the argument developed in section A.2, when the mirror carried by B' reflects the second pulse sent by A, the clock of B' displays the time $36\,\mathrm{s}$ but the clock of A already indicates $39\,\mathrm{s}$. Hence, for A, ***the moving clock*** carried by B' ***is running slow***. The ratio between the time intervals measured by the clock at rest and by the clock in motion, known as ***Lorentz*** γ ***factor*** is

$$\gamma = \frac{39\,\mathrm{s}}{36\,\mathrm{s}} = \frac{13}{12}\,. \tag{A.10}$$

As demonstrated in the previous section, in this example, the velocity of observer B' with respect to observer A is $v = 5/13 \times c$. The relationship between the Lorentz factor γ and the relative velocity v, deduced in Chapter 5, is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{A.11}$$

The reader may easily verify that this relation is satisfied in the present example, merely by checking that

$$\frac{1}{\sqrt{1 - (5/13)^2}} = \frac{13}{12}\,. \tag{A.12}$$

A.4 Length contraction - example

As explained in chapter 6, one considers a platform whose proper (that is, at rest) length is given and one asks what length will be attributed to this same platform by an observer for whom it is moving with a given velocity.

In this numerical example, it will be assumed that the proper length of the platform is[1]

$$L_0 = 26\,\mathrm{s{-}luz} \tag{A.13}$$

and that the platform moves with velocity

$$v = \frac{5}{13}\,c \tag{A.14}$$

which, as calculated in Sections A.2 and A.3, corresponds to the Bondi factor

$$k = \frac{3}{2} \tag{A.15}$$

and to the Lorentz factor

$$\gamma = \frac{13}{12}\,. \tag{A.16}$$

Denote by A the observer who performs the measurement of the length of the moving platform. The procedure utilized by A consists in sending light pulses to be reflected simultaneously (from his point of view) by mirrors carried by observers B' and C' who are located each at one extremity of the platform and are standing still on it.

The diagram of Figure A.3 depicts the situation from the point of view of observer A. Observer B' passes by observer A first; some time later, observer C' passes by observer A

[1] This is quite a respectable length for a platform, about 20 times the distance from the Earth to the Moon!

and one may assume that when observers A and C' meet, both reset their clocks to zero. Hence, at that event, the clock of A marks $0\,\mathrm{s}$ and the clock of C' also marks $0\,\mathrm{s}$ (event *O*) .

In order to develop the argument based on the use of the Bondi factor, it is convenient to suppose that when C' passes by A, the latter sends a first light pulse which is reflected instantaneously by the mirror of C' and immediately comes back to A. At that same event, observer A also sends a pulse to observer B' (event *I*, which coincides with event *O*).

Some time later, observer A sends another light pulse to be reflected by the mirror of C'. As discussed in Chapter 6, observer A must choose the emission time of this pulse in such a manner that, from the point of view of A, the reflection by the mirror of C' happens simultaneously with the reflection of the pulse that was sent toward the mirror of B'. Knowing the values of the proper length of the platform and of the Bondi factor, observer A can calculate this time [see equation (6.11), remembering that $L' = L_0$] as

$$\frac{2\,k\,L_0}{(1+k^2)\,c} = \frac{2\times\frac{3}{2}}{1+(\frac{3}{2})^2}\times\frac{26\,\text{light-s}}{1\,\text{light-s/s}} = 24\,\mathrm{s}\;. \tag{A.17}$$

That is, A should send the second pulse to C' when his clock marks $24\,\mathrm{s}$ (event *J*).

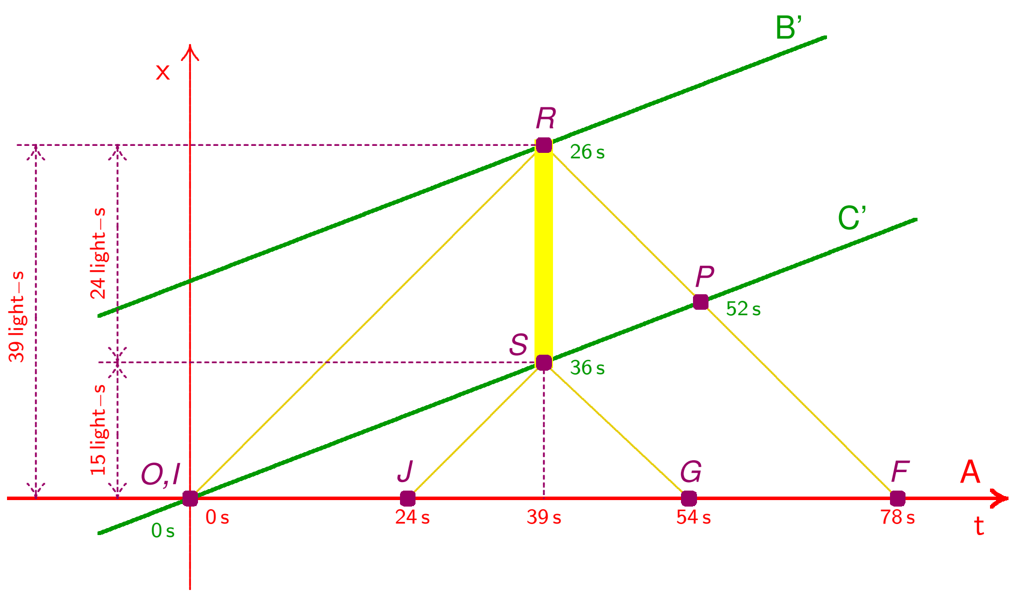

Figure A.3: *Minkowski diagram showing the situation considered in the analysis of the length contraction of a platform. The diagram's axes refer to the referential of observer A for whom the platform is in motion. The world lines of two observers B' and C' located at the platform's extremities are drawn. For A, the two light pulses he sends are reflected simultaneously by mirrors carried by B' and C'. The graphic determination of the length attributed to the platform by observer A is made explicit.*

Since the emission interval of the two pulses sent by A to C' was $24\,\mathrm{s}$, the reception (by C') interval is

$$\frac{3}{2}\times 24\,\mathrm{s} = 36\,\mathrm{s}\;. \tag{A.18}$$

Therefore, when C' receives the second pulse coming from A, his clock marks 36 s (event *S*). Since the reflection interval of the two pulses by the mirror of C' was 36 s, the interval of reception by A of the reflected pulses must be

$$\frac{3}{2} \times 36\,\mathrm{s} = 54\,\mathrm{s}\,. \tag{A.19}$$

The time indicated by the clock A when he receives the second pulse reflected by C' is then 54 s (event *G*).

By the constancy of light velocity, the time attributed by A to the event of reflection, by C', of the second pulse (event *S*) must be the median between the time of emission and the time of reception of the reflected pulse, that is,

$$\frac{24\,\mathrm{s} + 54\,\mathrm{s}}{2} = 39\,\mathrm{s}\,. \tag{A.20}$$

Since, during this time interval, observer C' moved away from A with velocity $\frac{5}{13}\mathrm{c}$, it is easily deduced that the reflection event *S* occurred at a distance from A given by:

$$\frac{5}{13} \times \frac{1\,\mathrm{light{-}s}}{\mathrm{s}} \times 39\,\mathrm{s} = 15\,\mathrm{light{-}s}\,. \tag{A.21}$$

Consider now the light pulse sent by A toward the mirror of B'. It was sent when C' passed by A and reset his clock to zero, therefore at 0 s, by the clocks of B' and C'. For these observers, it took

$$\frac{\mathrm{L}_0}{\mathrm{c}} = \frac{26\,\mathrm{light{-}s}}{\frac{1\,\mathrm{light{-}s}}{\mathrm{s}}} = 26\,\mathrm{s} \tag{A.22}$$

to reach B'; hence, when his mirror reflects this pulse, the clock of B' marks 26 s (event *R*). The reflected pulse takes the same time to get back to C' and, therefore, when it passes by C', the clock of C' marks 52 s (event *P*). By considering, together with this reflected pulse, the one which was reflected instantaneously when C' passed by A, one sees that the interval between the passings by C' of these two pulses was 52 s, by the clock of C'. So, the interval of reception by A was

$$\frac{3}{2} \times 52\,\mathrm{s} = 78\,\mathrm{s} \tag{A.23}$$

and, when observer A receives the pulse reflected by B', his clock marks 78 s (event *F*). For observer A, the event of reflection of the pulse by B' occurred at the median instant between the instant of emission and the instant of reception of the reflected pulse, hence at time

$$\frac{0\,\mathrm{s} + 78\,\mathrm{s}}{2} = 39\,\mathrm{s}\,. \tag{A.24}$$

This confirms then that the reflection events at the two extremities of the platform (events *R* and *S*) occurred simultaneously for observer A, as had to be the case if the whole operation was truly to correspond to a measurement, by A, of the platform's length. To complete the calculation of this length, it merely remains to determine the distance between observer A and the most distant extremity of the platform at the time of measurement. Since the light pulse reflected by B' took 39 s to reach B', this distance obviously is

$$\frac{1\,\mathrm{light{-}s}}{\mathrm{s}} \times 39\,\mathrm{s} = 39\,\mathrm{light{-}s}\,. \tag{A.25}$$

The length measured by A is the difference between the distances of the two extremities of the platform at the instant of measurement:

$$39\,\text{light-s} - 15\,\text{light-s} = 24\,\text{light-s}\,. \tag{A.26}$$

It can be immediately checked that this result is in agreement with the theory of length contraction [see relation (6.18)]:

$$\frac{L_0}{\gamma} = \frac{26\,\text{light-s}}{\frac{13}{12}} = 24\,\text{light-s}\,. \tag{A.27}$$

It is worthwhile emphasizing that the events *R* and *S* of reflection of the pulses at the platform's two extremities, although simultaneous for observer A – both occur when this observer's clock marks 39 s – are not simultaneous for observers B' and C'. Event *R* occurs when the clock of B' marks 26 s whereas event *S* occurs when the clock of C' (and therefore also the clock of B', since these two observers are at rest with respect to each other) marks 36 s. One has here an example of the ***relativity of simultaneity***, a noteworthy characteristics of Special Relativity: events which happen simultaneously, but at different locations, for a given observer, in general do not happen simultaneously for another observer who is moving with respect to the first.

A.5 Combination of velocities - example

Observer B' is moving with respect to observer A and observer C" is moving in the same direction with respect to observer B'.

Let $k = 3/2$ be the Bondi factor relating B' to A, and $k' = 4/3$ be the Bondi factor relating C" to B'.

Observer A emits light pulses at 6 s intervals. These pulses are received by B' at intervals of

$$6\,\text{s} \times \frac{3}{2} = 9\,\text{s}\,. \tag{A.28}$$

Every time B' receives a pulse coming from A, he also emits a pulse. These pulses, emitted by B' at 9 s intervals, are received by C" at intervals of

$$9\,\text{s} \times \frac{4}{3} = 12\,\text{s}\,. \tag{A.29}$$

The diagram of Figure A.4 illustrates the situation from the point of view of observer A.

Since the pulses coming from A travel abreast with the pulses emitted by B', they also are received by C" at 12 s intervals. Hence, the Bondi factor relating C" to A is

$$K = \frac{12\,\text{s}}{6\,\text{s}} = 2\,. \tag{A.30}$$

This result obviously illustrates the multiplicative combination of Bondi factors, for

$$\frac{4}{3} \times \frac{3}{2} = 2\,. \tag{A.31}$$

As discussed in Chapter 4, the relative velocity v is given in terms of the k factor by [see relation (4.8)]

$$\mathrm{v} = \frac{\mathrm{k}^2 - 1}{\mathrm{k}^2 + 1}\,\mathrm{c}\,. \tag{A.32}$$

Hence, the velocity of B’ with respect to A is

$$\mathrm{v} = \frac{(3/2)^2 - 1}{(3/2)^2 + 1}\,\mathrm{c} = \frac{5}{13}\,\mathrm{c} \tag{A.33}$$

and the velocity of C” with respect to B’ is

$$\mathrm{v}' = \frac{(4/3)^2 - 1}{(4/3)^2 + 1}\,\mathrm{c} = \frac{7}{25}\,\mathrm{c}\,. \tag{A.34}$$

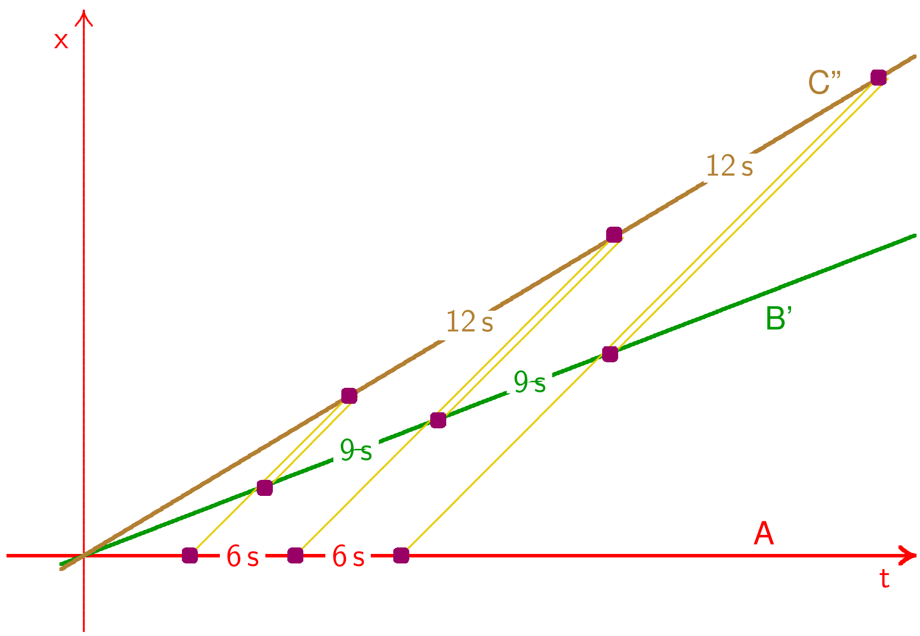

Figure A.4: *Minkowski diagram showing the situation considered in the analysis of the combination of Bondi factors relative to motions in the same direction. The diagram’s axes refer to the referential of observer A. The world lines of observers A, B’ and C”, and of the light pulses emitted at regular intervals by observers A and B’, are drawn. The intervals of emission and/or reception of the pulses, measured by the observer involved, are indicated.*

Using the result obtained in the combination of Bondi factors, one can also calculate the velocity V of C” with respect to A:

$$\mathrm{V} = \frac{2^2 - 1}{2^2 + 1}\,\mathrm{c} = \frac{3}{5}\,\mathrm{c}\,. \tag{A.35}$$

The combination law for collinear velocities, that is [see formula (7.8)]

$$\mathrm{V} = \frac{\mathrm{v}' + \mathrm{v}}{1 + \frac{\mathrm{v}'\mathrm{v}}{\mathrm{c}^2}}\,, \tag{A.36}$$

corresponds therefore to the arithmetic relation

$$\frac{\frac{7}{25}\,c + \frac{5}{13}\,c}{1 + \frac{1}{c^2} \times \frac{7}{25}\,c \times \frac{5}{13}\,c} = \frac{3}{5}\,c\,, \tag{A.37}$$

which is easily checked.

It is worth commenting that, in the kinematics of Galileo and Newton, in which the combination law for collinear velocities is the simple sum, the result for the velocity of C” with respect to A would be

$$V_{GN} = v' + v = \frac{7}{25}\,c + \frac{5}{13}\,c = \frac{216}{325}\,c = 0.665\,c\,. \tag{A.38}$$

It can be seen that, in the example considered here, this value differs from the result given by Einstein’s Relativity $V = 0.6\,c$ by about 10 %. If the involved velocities v and v' were both very close to the speed of light, the difference would be almost a factor of 2, since V_{GN} would be nearly $2\,c$, whereas V would remain inferior to c.

A.6 Twin paradox - example

On planet Earth, considered here as an inertial referential, lives an observer A who is the father of twins John and Peter. One day, when John is 5 years old and Peter obviously is also 5 years old, a spaceship piloted by observer B’ passes by the Earth.

Fascinated by spatial adventures and by Special Relativity, A decides to entrust one of the twins, John, to the traveler B’, whereas the other twin, Peter, stays home.

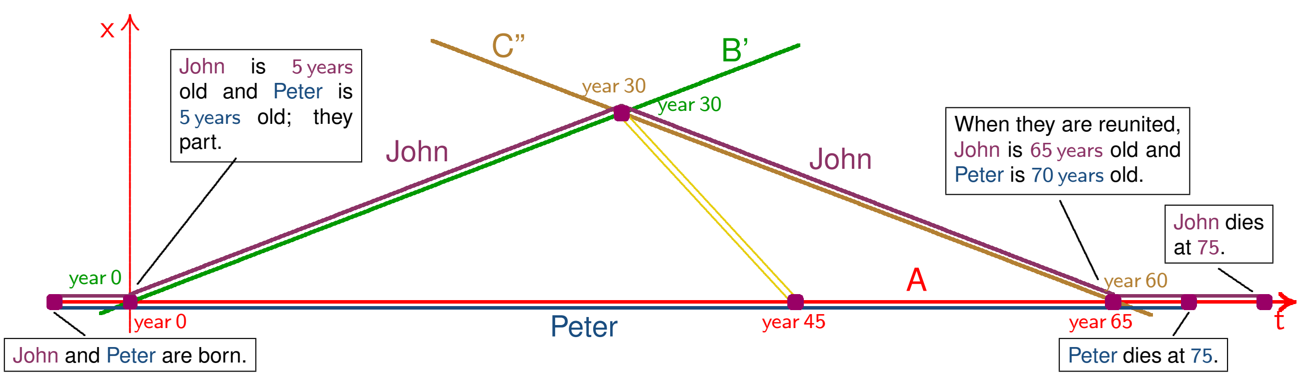

Figure A.5: *Minkowski diagram showing the situation considered as illustration of the twin paradox. The diagram axes refer to the referential of observer A. The world lines of observers A, B’ and C”, and of the light pulses emitted by observers B’ and C” when they pass by each other, are drawn. Also shown are the world lines of the twins, John and Peter, as well as descriptions of a few important events in their lives.*

Some years later, the spaceship commanded by B’ encounters another spaceship, piloted by observer C”, traveling in the opposite direction, that is, toward the Earth where A lives, with the same speed with which B’ is moving away from Earth. Wistful for home, John resolves to take advantage of the opportunity and transfers himself to the spaceship of C”.

When this spaceship reaches the Earth, John is reunited with his twin brother Peter and his father A.

The diagram of Figure A.5 illustrates the situation from the point of view of observer A. It is assumed that the velocity of B' with respect to A is $\frac{5}{13}\,c$, which corresponds to the Bondi factor $k = 3/2$ and the Lorentz factor $\gamma = 13/12$ [see sections A.2 and A.3].

In order to simplify the analysis, it may be assumed that, when the observers A and B' pass by each other, they both set their clocks to zero. For the development of the argument based on the use of the Bondi factor, it is convenient to suppose that, at this event, observer B' sends a first light pulse to A, who receives it instantaneously, since both observers are at the same place.

Let us assume that 30 years later (by his own clock), B' passes by C", who is traveling in the opposite direction, moving toward A with the same velocity $\frac{5}{13}\,c$. The Bondi factor relating C" to A is therefore $1/k = 2/3$.

When C" passes by B', he adjusts his clock to that of B', which reads 30 years at that instant. Since John traveled with B', he aged 30 years during this part of his trip and is now

$$5\text{ years} + 30\text{ years} = 35\text{ years} \tag{A.39}$$

old. At that moment, he leaves the spaceship of B' and begins his trip back toward home in the spaceship of C".

When B' passes by C", he sends a second pulse to A. Since the emission interval, for B', of his two pulses, was 30 years, the reception interval of these pulses by A, is

$$\frac{3}{2} \times 30\text{ years} = 45\text{ years}\ , \tag{A.40}$$

which is to say that, when A receives the second pulse from B', his clock indicates 45 years.

When C" passes by B', he too sends a light pulse to A, who receives it also when his clock marks 45 years.

Since C" is moving toward A with the same speed with which B' is moving away, he takes 30 years, by his own clock, to reach A. Given that John accompanies C", he ages 30 years during this part of his trip. Hence, his total aging during his trip is

$$30\text{ years} + 30\text{ years} = 60\text{ years} \tag{A.41}$$

and when he meets again with his father A and his twin brother Peter, he is

$$5\text{ years} + 60\text{ years} = 65\text{ years} \tag{A.42}$$

old.

One may imagine that, when C" reaches A, he sends a second pulse to A, who receives it instantaneously since both observers are at the same spot. By the definition of the Bondi factor, the interval of reception by A of the two pulses emitted by C" is

$$\frac{2}{3} \times 30\text{ years} = 20\text{ years}\ . \tag{A.43}$$

From these results, one deduces that, when he meets with C", the clock of A marks

$$45\text{ years} + 20\text{ years} = 65\text{ years}\ . \tag{A.44}$$

Since Peter remained together with A the whole time, he aged 65 years and is therefore

$$5\ \text{years} + 65\ \text{years} = 70\ \text{years} \tag{A.45}$$

old when he meets C”, an event that corresponds also to his re-encounter with his twin brother John.

One concludes that, when the twins are finally reunited, John is 65 years old and Peter is 70 years old. That is to say: ***the twin who went away and came back aged more slowly than the other, who stayed home***.

It is easy to check that the ratio between the aging of John, who traveled, and the aging of Peter, who stayed put, is given by the inverse of the Lorentz factor, as formula (8.8) predicts:

$$\frac{60\ \text{years}}{65\ \text{years}} = \frac{12}{13} = \frac{1}{\gamma}\,. \tag{A.46}$$

Given that the twins are biologically identical, it is plausible that both die when they reach the same biological age, 75 years, for example. However, when Peter passes away, at age 75, John will be only 70 years old and he will spend the last 5 years of his life mourning the loss of his brother Peter.

Bibliography

[] **Hermann Bondi**, *Relativity and Common Sense: a new Approach to Einstein*, Dover Publications Inc, New York, USA, 1980.